Optimal Contracting under Moral Hazard

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Market interactions

Transactions between agents are motivated by exchanging risks :

- different expositions to risk
- different appetite for risk
- different hedging purposes

Jean Tirole : interactions, including Industrial organization, are dictated by incentives

Goal : simple modeling for market interactions in the context of delegation

Risk Sharing

- Principal draws utility from **output** :

$$X^a := a + N \quad \text{for some r.v.} \quad N \sim \mathcal{N}(a, \sigma), \quad \sigma > 0 \;\; \text{given}$$

- Management of ouput delegated to Agent :
  - receives the random amount ξ : **the contract**
  - devotes **effort** a inducing $X^a \sim \mathcal{N}(a, \sigma)$
  - cost of effort $c(a)$
- Principal chooses contract and effort :

$$\max_{\xi, a} \mathbb{E}\big[U_P(X^a - \xi)\big] + \lambda \mathbb{E}\big[U_A\big(\xi - c(a)\big)\big]$$

i.e. maximization of joint welfare by social planner (Pareto optimal)

Borch Rule for Risk-Sharing

- First order condition in $\xi \Longrightarrow$ Borch rule :

$$\frac{U_P'(X^{\hat{a}} - \hat{\xi})}{U_A'\big(\hat{\xi} - c(\hat{a})\big)} \;=\; \lambda$$

- First order condition in $a \Longrightarrow$

$$c'(\hat{a}) \;=\; \frac{U_P'(X^{\hat{a}} - \hat{\xi})}{\lambda U_A'\big(\hat{\xi} - c(\hat{a})\big)} = 1$$

Typically, $c : \mathbb{R}_+ \longrightarrow \mathbb{R}$ increasing, strictly convex,

Optimal effort : $\quad \hat{a} := (c')^{-1}(1)$

Risk-Sharing under exponential utilities

Let $U_A(x) := -\frac{1}{\gamma_A}e^{-\gamma_A x}$ and $U_P(x) := -\frac{1}{\gamma_P}e^{-\gamma_P x}$

Then, Borch rule reduces to :

$$\hat{\xi} \quad := \quad \frac{\gamma_P}{\gamma_A + \gamma_P} X + \frac{\gamma_A c\big((c')^{-1}(1)\big) + \log \lambda}{\gamma_A + \gamma_P}$$

i.e. optimal contract = constant payment + proportion of output

Does not reflect reality in market interactions :

- Agent devotes **constant effort !**
- Contract proportional to output, **regardless of risk !**

Moral hazard

• Adam Smith (1723-1790) identified moral hazard as a major risk in economics :

Situation where an agent may benefit from an action whose cost is supported by others

Should not count on agents' morality...

Principal-Agent Problem

- Principal does not observe Agent's effort, then contract

$$\xi = \xi(X^a)$$

- Agent determines optimal effort by

$$V_A(\xi) := \max_a \mathbb{E} U_A\big(\xi(X^a) - c(a)\big) \implies \hat{a}(\xi)$$

- Principal chooses optimal contract by solving

$$\max_\xi \mathbb{E} U_P\big(X^{\hat{a}(\xi)} - \xi(X^{\hat{a}(\xi)})\big) \quad \text{under constraint} \quad V_A(\xi) \geq R$$

Non-zero sum Stackelberg game

Difficult to solve, and so restrict to affine contracts...

Affine contract under exponential utilities and quadratic cost

Let $c(a) := \frac{1}{2} c_0 a^2$, and find the best contract of the form

$$\xi(x) = k_0 + k_1 x$$

• From the agent's problem, we directly compute that

$$\hat{a}(\xi) = \frac{k_1}{c_0}$$

• Saturating the constrain in the Principal problem $V_A(\xi) = R$ leads to

$$k_0 = \frac{k_1^2}{2}\left(\gamma_A \sigma^2 - \frac{1}{2c_0}\right) - \frac{\log R}{\gamma_A}$$

$\Longrightarrow$ Principal's problem reduces to maximization over k_1...

$$\hat{k}_1 = \frac{\frac{1}{c_0\sigma^2} + \gamma_P}{\frac{1}{c_0\sigma^2} + \gamma_P + \gamma_A}$$

Comments on Second Best

- Difficult to solve in the present one-period setting !
- Restriction to affine contracts :

how good is this class ?

In particular, we may be pushing Agent to take more risk !

Our Main Objective

This problem is more accessible in continuous time

Started by Holmström & Milgrom 1985...

Nobel Prize 2016 winners : Oliver Hart and Bengt Holmström

Holmström & Milgrom 1985

Output process with effort α :

$$dX^{\alpha_t} = \alpha_t dt + \sigma dW_t$$

Agent solves

$$\max_{\alpha} \mathbb{E}\Big[\xi(X^{\alpha}) - \frac{1}{2}c_0 \int_0^T |\alpha_t|^2 dt\Big] \implies \hat{\alpha}(\xi)$$

Principal solves

$$\max_{\xi} \mathbb{E}\Big[U_P(X^{\hat{\alpha}(\xi)}{}_T - \xi(X^{\hat{\alpha}(\xi)})\Big]$$

- Non-zero sum stochastic differential game

Moral hazard : financial regulation

⟹ Remedy : regulation, compensation indexed by risks

Fund managers compensation under moral hazard

Fund managers portfolio value for effort $\nu = (\alpha, \pi)$

$$dX^{\nu}{}_t = \pi_t \cdot (\alpha_t dt + dW_t)$$

Manager's problem

$$\sup_{\nu=(\alpha,\pi)} \mathbb{E}\Big[\xi - \int_0^T \big(c_0\alpha_t^2 - c_1|\pi_t|^2\big)\, dt\Big] \implies \hat{\nu}(\xi)$$

Principal problem :

$$\sup_{\xi(.)} \mathbb{E}\big[U\big(X^{\hat{\nu}(\xi)}{}_T - \xi(X^{\hat{\nu}(\xi)})\big)\big]$$

Quadratic variation and riskiness of fund management

- X_t is the value of the fund at time t

- Principal only observe the relized gains $\{X_t(\omega), t \in [0, T]\}$, and has no access to the distribution of X

- Quadratic variation, also called realized variance :

$$\langle X \rangle_t \quad := \quad \text{“} \lim_{\Delta t \searrow 0} \text{”} \sum_{t_i \le t} \left| X_{t_i} - X_{t_{i-1}} \right|^2$$

measures the risk induced by the fund manager

Optimal fund manager compensation

Our main result characterizes the optimal contract as

$$\hat{\xi} = \int_0^T \hat{Z}_t \cdot dX_t + \frac{1}{2}\hat{\Gamma}_t : d\langle X\rangle_t - H(\hat{Z}_t, \hat{\Gamma}_t)dt$$

where $\langle X\rangle$ is the quadratic variation of the output X

$$H(z,\gamma) \quad := \quad \sup_{\pi,\alpha}\left\{\pi\cdot\alpha z + \frac{1}{2}|\pi|^2\gamma - c_0|\alpha|^2 - c_1|\pi|^2\right\}$$

and $\hat{Z}$, $\hat{\Gamma}$ are determined by means of a HJB equation...

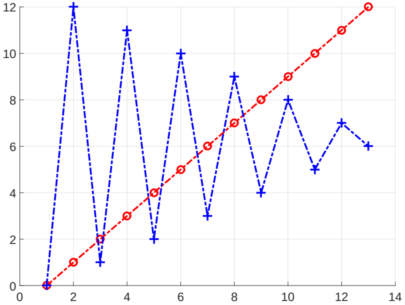
Moral hazard : electricity tarification

Electricity generation by solar / windfall technologies is volatile

$\Longrightarrow$ Although consumers demand is quite predictable, adjustment of electricity demand by classical generation means inherits important variability

$\Longrightarrow$ Path-dependent tarification

Volatility of consumption



Total consumption of X = Total consumption of X

$\langle X \rangle = 1^2 + \cdots + 1^2 = 12 \qquad \langle X \rangle = 12^2 + 11^2 + \cdots + 1^2 = 650$

Electricity tarification under moral hazard

Electric power demand in excess to a predictable reference pattern :

$$dX^{\nu}{}_t = -\alpha_t dt + \sigma \beta_t dW_t, \quad \alpha_t \geq 0, \quad \beta_t \in (0,1]$$

Consumer problem

$$\sup_{\nu=(\alpha,\beta)} \mathbb{E}\Big[-\xi + \int_0^T \big(u(X^{\nu}{}_t) - c(\nu_t) \big) dt \Big] \implies \hat{\nu}(\xi)$$

$u(x) := -e^{-\eta x}$ and $c(a,b) := c_0 a^2 + c_1 b^{-2}$

Producer problem :

$$\sup_{\xi(.)} \mathbb{E}\Big[\xi(X^{\hat{\nu}(\xi)}) + \int_0^T \pi(X^{\hat{\nu}(\xi)}{}_t) dt - q \langle X^{\hat{\nu}(\xi)} \rangle_t \Big]$$

Optimal tarification

Our main result characterizes the optimal tarification as

$$\hat{\xi} = \int_0^T \hat{Z}_t \cdot dX_t + \frac{1}{2}\hat{\Gamma}_t : d\langle X \rangle_t - H(\hat{Z}_t, \hat{\Gamma}_t)dt$$

where $\langle X \rangle$ is the quadratic variation of the output X

$$H(z,\gamma) \quad := \quad \sup_{\alpha,\beta} \Big\{ -\alpha z + \frac{1}{2}\sigma^2\beta^2\gamma - c_0|\alpha|^2 - c_1|\beta|^{-2} \Big\}$$

and $\hat{Z}$, $\hat{\Gamma}$ are determined by means of a HJB equation...

Empirical results

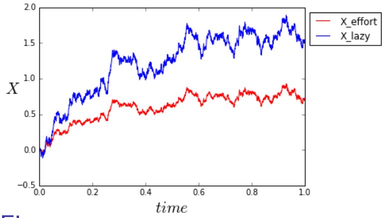


Figure – A sample path of daily consumption : incited, non incited

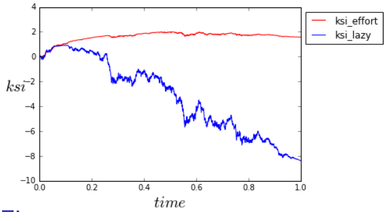


Figure – A sample path of incentive premium : incited, non incited

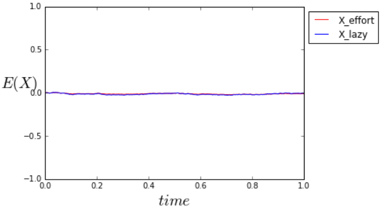


Figure – Average daily consumption : incited, non incited

Approximate contract

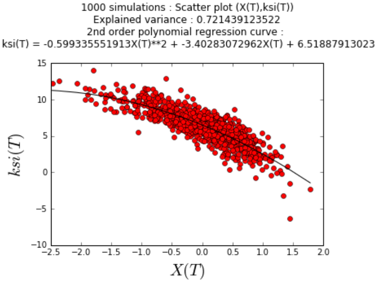
Recall

$$\hat{\xi} = \int_0^T \hat{Z}_t \cdot dX_t + \frac{1}{2}\hat{\Gamma}_t : d\langle X \rangle_t - H(\hat{Z}_t, \hat{\Gamma}_t)dt$$

$\Longrightarrow$ Simulate N paths of X, and compute the corresponding $\hat{\xi}$'s

$\Longrightarrow$ Regression of $\hat{\xi}$

- on X_T
- on (X_T, X_T^2)
- on $(X_T, X_T^2, X_{T/2}, X_{T/2}^2)$
  $\Longrightarrow R^2 \sim .95\%$



Principal-Agent problem : general formulation

- Agent solves the control problem :

$$V_0^A(\xi) := \sup_{\nu=(\alpha,\beta)} \mathbb{E}\Big[K_T\xi - \int_0^T K_t c_t(\nu_t)dt\Big]$$

where $K_t = e^{-\int_0^t k_s^\nu ds}$ and Output process :

$$dX = \sigma_t(X,\beta_t)\big[\lambda_t(X,\alpha_t)dt + dW_t\big]$$

- Principal solves the optimization problem

$$V_0^P := \sup_{\xi\in\Xi_R} \mathbb{E}\Big[K_T^{\nu^\star} U(\ell(X^{\nu^\star}) - \xi(X^{\nu^\star}))\Big]$$

where Ξ_R : $\xi(X_.)$, such that $V_0^A(\xi) \geq R$

Path-dependent Hamiltonian

- Path-dependent Hamiltonian for the Agent problem :

$$H_t(,y,z,\gamma) \quad := \quad \sup_{a,b}\Big\{\sigma_t(,a)\lambda_t(,b)\cdot z + \tfrac{1}{2}\sigma_t\sigma_t^\top(,a):\gamma \\ -k_t(,a,b)y - c_t(,a,b)\Big\}$$

- For $Y_0 \in \mathbb{R}$ and $Z,\Gamma\ \mathbb{F}^X - \text{prog meas}$, define

$$dX_t = \nabla_z H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)dt + \big\{2\nabla_\gamma H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)\big\}^{\frac{1}{2}} dW_t$$

$$dY_t^{Z,\Gamma} = Z_t \cdot dX_t + \tfrac{1}{2}\Gamma_t : d\langle X\rangle_t - H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)dt$$

$$V_0(X_0, Y_0) \quad := \quad \sup_{Z,\Gamma} \mathbb{E}\Big[U\big(\ell(X) - Y_T^{Z,\Gamma}\big)\Big]$$

Main result

Theorem

We have

$$V_0^P = \sup_{Y_0 \geq R} V_0(X_0, Y_0)$$

Given maximizer $Y_0^\star$, *the corresponding optimal controls* $(Z^\star, \Gamma^\star)$ *induce an optimal contract*

$$\xi^\star = Y_T^{Z^\star,\Gamma^\star} = Y_0^\star + Z_t^\star \cdot dX_t + \frac{1}{2}\Gamma_t^\star : d\langle X\rangle_t - H_t(X, Y_t^{Z^\star,\Gamma^\star}, Z_t^\star, \Gamma_t^\star)dt$$

On the function V

$\bar{X} := (X, Y)$ satisfies $d\bar{X}_t = \bar{\mu}(\bar{X}_t, Z_t, \Gamma_t)dt + \bar{\sigma}(\bar{X}_t, Z_t, \Gamma_t)dW_t$:

$$dX_t = \nabla_z H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)dt + \left\{2\nabla_\gamma H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)\right\}^{\frac{1}{2}} dW_t$$
$$dY_t^{Z,\Gamma} = Z_t \cdot dX_t + \tfrac{1}{2}\Gamma_t : d\langle X\rangle_t - H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)dt$$

and recall $V_0(\bar{X}_0) := \sup_{Z,\Gamma} \mathbb{E}\left[U(\ell(X_T) - Y_T^{Z,\Gamma})\right]$

$V_0(\bar{X}_0) = V(0, \bar{X}_0)$; V solution of Hamilton-Jacobi-Bellman eq.

$$\partial_t V_0 + \sup_{z,\gamma}\left\{\bar{\mu}(.,z,\gamma)DV_0 + \tfrac{1}{2}\overline{\sigma\sigma}^\top(.,z,\gamma) : D^2 V_0\right]\Big\} = 0$$
$$V_0(T, x, y) = U(\ell(x) - y)$$

Extensions

- Limited liability : add state constraint $Y \geq 0$
- Optimal contract termination (by Agent and/or Principal) : add optimal stopping
- Infinite horizon
- Heterogeneous agents
- Mean field interaction between agents